Education

Unit Decay: A Clinically Oriented Perspective on Teaching Exponential Decay

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The concept of exponential phenomena can be difficult. Exponential processes in nuclear medicine can be simplified by using a new concept, the unit decay constant (UDC). The UDC is the dimensionless percentage expressed as a fraction that is found to be remaining of an initial value. V_0***, after a "unit" length of measure, such as time, has elapsed. This change results in a new value,*** V_1***, and the ratio of*** V_1/V_0 ***is the UDC. This paper demonstrates the development of this constant and its use in the clinical setting. The UDC provides a different perspective of the exponential process that is easily understood and more intuitive than exponentiation in base "e." In addition, the UDC allows for rapid calculations of decay and attenuation using only simple multiplication in some cases.***

Processes that follow the laws of exponential mathematics are encountered in nuclear medicine and are taught in nuclear medicine training programs (*1,2*). However, some physicians and technologists do not fully appreciate the properties of the exponentially shaped curve or its mathematical application for calculating loss of radioactivity and photon flux. For those individuals who have a good understanding of mathematics, these concepts may not prove to be difficult. However, others that do not have a good understanding of exponential mathematics, may find these concepts difficult to understand and apply. Difficulties most often arise when one attempts to translate basic physical phenomena into usable data for any given clinical situation. The concept of a number, represented by the letter "e," the natural exponential base, being raised to a negative power of λt or μX, can clearly test one's mathematical abilities. Another approach for handling exponential applications in nuclear medicine is presented. The traditional method of calculating exponential processes is contrasted with the technique of unit decay for radioactive disintegration and photon attenuation. The unit decay method permits these calculations to be performed in a simplified manner, which is more easily understood by persons unfamiliar with exponential mathematics.

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TRADITIONAL CONCEPTS OF EXPONENTIAL MATHEMATICS

Radioactive disintegration illustrates an exponential process and is used as an example. The process of radioactive decay is a spontaneous event, and for each radioactive element, a constant percentage of all of the atoms present will disintegrate over a given unit of time. According to the observed laws leading to exponential decay, the rate of decline in the number of radioactive atoms depends upon: (a) N(t), the actual number of radioatoms, N, present at any instantaneous moment in time, t; and (b) the fraction of atoms disintegrating during that infinitesimally small amount of time (decay percentage as a fraction per unit of time or λ). The value of λ varies depending upon the radionuclide being studied. Thus, the familiar equation:

$$dN/dt = -\lambda N(t). \qquad \text{Eq. 1}$$

Equation 1 is the basis for the appearance of the familiar exponential curve (see also Eq. 3). If radioactive decay followed linear mathematics and depended only upon a fixed value change per unit of time irrespective of the total number of atoms present at any time (t) (i.e., $N(t) = -\lambda N_0 t + N_0$), then the decay curve would be a straight line leading to zero at a specified moment in time (Fig. 1). In this latter case, the time at which zero atoms are reached occurs at the "mean life," which is $1/\lambda$ or $1.44t_{1/2}$.

The quantity described by the format of dN/dt is expressed as decays, or atoms changing, per second, minute, etc. This represents the *activity* of any sample.

Activity in Curies is related to the number of atoms present and its decay constant, λ, in the following manner:

$$A = \lambda N / (3.7 \times 10^{10}), \qquad \text{Eq. 2}$$

where A is in Curies (Ci), λ is a dimensionless fraction per unit time, N is in atoms, and there are 3.7×10^{10} Bq (or disintegrations/second) per Ci.

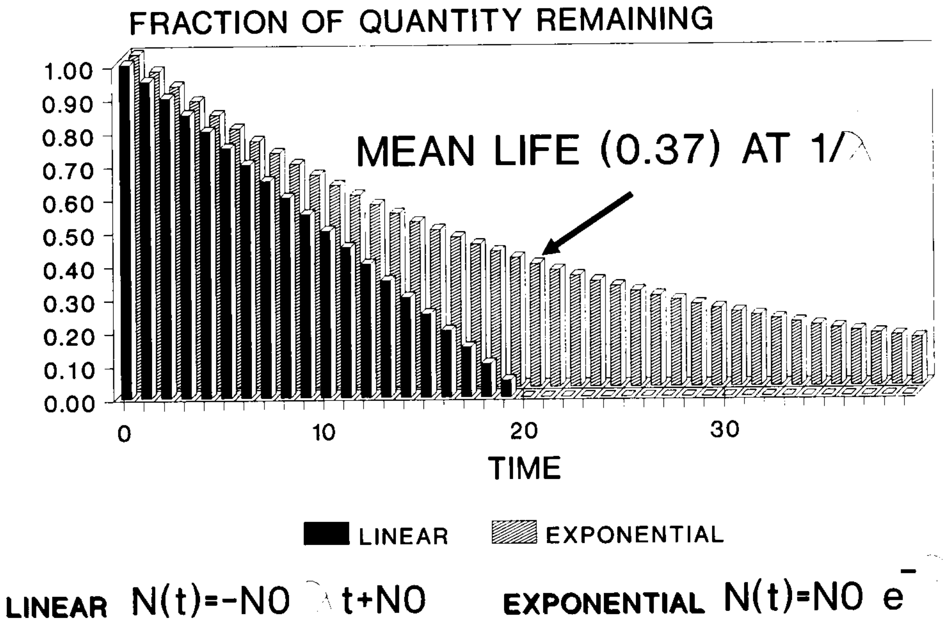

FIG. 1. Comparison of exponential decay against linear decay with both curves starting with the same initial number of atoms.

Starting with Eq. 1, further mathematical manipulations utilizing integration will solve Eq. 1 for N, and results in yet another familiar equation as follows:

$$N(t) = N_0e^{-\lambda t}, \qquad \text{Eq. 3}$$

where $N(0) = N_0$ is the number of atoms at the starting point $(t = 0)$.

Given that $A(t) = \lambda N(t)$ and that $N(t) = N_0e^{-\lambda t}$, it follows that by substituting for N in the first equation, we get:

$$A(t) = \lambda N_0e^{-\lambda t} \text{ and since } A_0 = \lambda N_0$$

then

$$A\ (t) = A_0e^{-t}, \qquad \text{Eq. 4}$$

where A(t) is the activity present at time t, and $A(0) = A_0$ is the activity at the starting point $(t = 0)$.

The concept of the half-life of an isotope is one that can be understood fairly easily, and it provides valuable and helpful information for nuclear medicine (Table 1). The half-life of a radionuclide is the amount of time required for the isotope to decay or to reduce its initial activity, A_0, to 50% of its activity level, that is $A_0/2$. From Eq. 4, one can derive that the time for a half-life to occur is equal to 0.693 divided by λ or:

$$T_{1/2} = 0.693/\lambda. \qquad \text{Eq. 5}$$

These five equations represent the basic mathematics involved in dealing with exponential phenomena of radioactive decay. By substituting known data into these equations, solutions can be calculated. However, it can be difficult to perform these mathematical calculations. The unit decay method provides a simpler means for performing these calculations. Exponentiation with this method requires the use of a calculator with an e^x or y^x function, a slide rule, or log tables.

TABLE 1. UDC Values

Radioactive decay			
Agent	**Half-life**	**Unit**	**UDC**
^{99m}Tc	6.03 hr	1 hr	0.89/hr remaining
^{201}Tl	72.0 hr	1 hr	0.99/hr remaining
^{123}I	13.0 hr	1 hr	0.95/hr remaining
^{111}In	67.0 hr	1 day	0.78/day remaining
^{67}Ga	78.1 hr	1 day	0.81/day remaining
^{131}I	8.06 days	1 day	0.92/day remaining

Photon attenuation				
Agent	**Half-value layer**	**Unit**	**Attenuator**	**UDC**
^{99m}Tc	4.59 cm	1 cm	soft tissue	0.860/cm transmitted
^{99m}Tc	0.25 mm	1 mm	lead shield	0.0625/mm transmitted
^{131}I	3.00 mm	1 mm	lead shield	0.794/mm transmitted

UNIT DECAY CONSTANT

Definition

In an effort to aid in the understanding of exponential decay and to simplify the mathematical calculations, the concept of the unit decay constant (UDC) has arisen as a clinically applicable method of handling exponential phenomena. The UDC is the dimensionless percentage expressed as a fraction that is found to be remaining of an initial value, V_0, after a "unit" length of measure, such as an integral quantity of time or distance, has elapsed or has been traversed, resulting in a new value, V_1, where:

$$V_1 = V_0(e^{-c \times \text{unit}})$$

or

$$V_1 = V_0(\text{UDC})^{\text{unit}}.$$

Therefore, the UDC is the ratio of V_1/V_0, given that the phenomenon being observed behaves in an exponential manner ($V_1/V_0 = e^{-c \times \text{unit}}$). With this assumption, mathematically, there will be a fractional drop of a constant amount, the UDC, for each fully completed integral unit of measurement as defined by the selected unit's value. The unit is the determining factor for the initial calculation of the UDC and should be selected for the appropriate clinical setting in which the constant is to be applied to calculate corrected or expected values.
Thus:

$$\text{UDC} = V_1/V_0,$$

where V_1 is calculated from V_0 after a single unit amount of measurement has occurred:

$$V_1 = V_0(e^{-c})^{(1\ \text{"unit"})} \qquad \text{Eq. 6}$$

or

$$V_1 = V_0 e^{-\lambda \times \text{conversion factors/"unit"} \times (1\ \text{"unit"})}.$$

If V_0 equals unity (i.e., $V_0 = 1$), then the value of V_1 will equal the "UDC" after a "unit" amount of measurement has occurred:

$$\text{UDC} = V_1 = 1 \times e^{-c \times (1\ \text{"unit"})}$$

or

$$V_1 = \text{UDC}(1\ \text{"unit"}) = \text{UDC}.$$

In Eq. 6, "unit" is equal to the physical unit measurement value (1 sec, 1 mo, 1 Å, 1 m, etc.), and c is equal to the percentage expressed as a fraction per unit measurement, λ, corrected to match the dimensions of the "unit" (i.e., if λ is a fraction per second and "unit" is picked to be 3 hr, then c = λ/sec × 60 sec/min × 60 min/hr × 3 hr/"unit"; or c = λ/sec × 10800 sec/"unit"; or c = λ x 10800/"unit"). The UDC = $e^{-\lambda \times \text{conversion factors/"unit"} \times (1\ \text{"unit"})}$ and then applied becomes:

$$V_1 = V_0 \text{UDC}^{\text{number of "units"}}.$$

We will apply this concept to solve selected problems to demonstrate once again how to derive the UDC and then apply it for rapid, clinically useful calculations.

Nuclear Medicine Applications

The UDC, once calculated, can be used to solve all possible presentations of the simple exponential phenomenon, as long as it is utilized in the units from which it was derived (Tables 2 and 3). Other units can be substituted if the appropriate conversion factor is applied to convert the new unit into terms consistent with the original unit. That is, if one derives the UDC for the unit being one day, and then wishes to solve a problem that deals with 72 hr, a conversion factor of 24 hr/day is applied to the 72 hr to result in three full units of 1 day. To appreciate how this works, let us derive and use the UDC for ^{99m}Tc for a unit of 1 hr/1 unit. We will assume that $V_0 = 1$ and calculate the value of V_1, which will equal the UDC. We will start with the routine exponential decay formula as follows:

$$A(t) = A_0 e^{-0.693t/T_{1/2}} \qquad \text{Eq. 7}$$

Assigned Values: $T_{1/2} = 6.02$ hr
Unit = 1 hr/unit
$V_0 = 1$

by substitution:

$$V_1 = 1 \times e^{-0.693/6.02\ \text{hr} \times 1\ \text{hr/unit}}$$
$$V_1 = 1 \times e^{-0.115} \qquad \text{Eq. 8}$$
$$V_1 = 0.89.$$

Thus the UDC equals 0.89 or 89% remains per unit of 1 hr. That is to say, 89% of Tc-99m atoms "remain" after 1 unit of time, where one unit of time was defined to be one hour. After each UDC of time passes, 89% of the atoms present at the beginning of the time period will remain.

To the amount of activity remaining in 3 hr, we find that 3 hr of time translates to 3 units. Therefore, 3 UDCs are applied to the initial value:

$$A(3\ \text{hr}) = A(0) \times \text{UDC} \times \text{UDC} \times \text{UDC}$$
$$A(3\ \text{hr}) = 1 \times 0.89 \times 0.89 \times 0.89 \qquad \text{Eq. 9}$$
$$A(3\ \text{hr}) = 0.70 \text{ or } 70\%.$$

Equation 9 also may be expressed as:

$$A(X\ \text{units}) = A_0(\text{UDC})^{X\ \text{units}}$$
$$A(3\ \text{units}) = 1 \times (0.89)^3$$
$$A(3\ \text{hr}) = 0.70 \text{ or } 70\%.$$

TABLE 2. UDC Calculation for ^{99m}Tc

Calculate the amount of ^{99m}Tc activity remaining at 4 hr from an original sample of 28.0 mCi.

Traditional	UDC
$A(t) = A_0 e^{-\lambda t}$	UDC (^{99m}Tc) = 0.89/hr
A(4 hr) = 28 mCi × $e^{-(0.693/6.02) \times 4}$	A(4) = A_0 (UDC)^4
A(4 hr) = 28 mCi × $e^{-0.115 \times 4}$	A(4) = 28 mCi × (0.89)^4
A(4 hr) = 28 mCi × $e^{-0.46}$	A(4) = 28 mCi × (0.89 × 0.89 × 0.89 × 0.89)
A(4 hr) = 28 mCi × 1/$e^{0.46}$	A(4) = 28 mCi × (0.63)
A(4 hr) = 28 mCi × 1/1.584	A(4) = 17.6 mCi
A(4 hr) = 28 mCi × 0.631	
A(4 hr) = 17.6 mCi	

TABLE 3. UDC Calculation for ^{201}Tl

Calculate the amount of acitivty needed to deliver a 3.3-mCi dose of ^{201}Tl allowing for a delay in delivery of 5 hr.	
Traditional	**UDC**
$A(t) = A_0e^{-\lambda t}$	UDC (^{201}Tl) = 0.99/hr
$3.3\ mCi = A_0 \times e^{-(0.693/72) \times 5}$	$A(5) = A_0 \times (UDC)^5$
$3.3\ mCi = A_0 \times e^{-0.0096 \times 5}$	$3.3\ mCi = A_0 \times (0.99)^5$
$3.3\ mCi = A_0 \times e^{-0.048}$	$3.3\ mCi = A_0 \times (0.99 \times 0.99 \times 0.99 \times 0.99 \times 0.99)$
$3.3\ mCi = A_0 \times 1/e^{0.048}$	$3.3\ mCi = A_0 \times (0.95)$
$3.3\ mCi = A_0 \times 1/1.049$	$A_0 = 3.3\ mCi/0.95$
$3.3\ mCi = A_0 \times 0.95$	$A_0 = 3.47\ mCi$
$A_0 = 3.3\ mCi/0.95$	
$A_0 = 3.47\ mCi$	

The concept of the UDC can also be applied to the following example: knowing that there will be a 4-hr delay in delivery, compute the amount of ^{99m}Tc activity needed to be prepared in advance to obtain a 10-mCi dose. We want to compute A(o) from the UDC, knowing that A(4 hr) = 10 mCi, that is, at 4 units of decay time.

Therefore:

$$A(4\ hr) = A(0\ hr)(UDC)^{\#\,of\,units} \qquad \text{Eq. 10}$$

Substituting

$$10\ mCi = X\ mCi \times (0.89)^4$$

Rearranging to solve for X mCi:

$$\begin{aligned} X\ mCi &= 10\ mCi/(0.89)^4 \\ X\ mCi &= 10\ mCi/0.627 \qquad \text{Eq. 11} \\ X\ mCi &= 15.94\ mCi. \end{aligned}$$

Calculations utilizing whole numbers, as noted above, can be easily done with a basic calculator. If fractional numbers become involved, however, the mathematics may prove difficult without the aid of a somewhat advanced calculator. If the calculator has a y^x function, then almost any number of units or fractions of units can be easily calculated. In the above problem, for example, if instead of a 4-hr delay, we are confronted with a 3-hr and 45-min delay (i.e., 3.75 hr), the solution is not difficult to obtain with the aid of a y^x function. We would then have 3 + ¾'s or 3.75 "1 hour units" of decay. Rearranging Eq. 11 results in:

$$\begin{aligned} X\ mCi &= 10\ mCi/(0.89)^{3.75} \\ X\ mCi &= 10\ mCi/0.6459 \qquad \text{Eq. 12} \\ X\ mCi &= 15.48\ mCi. \end{aligned}$$

By using a simple conversion factor (CF) and the y^x function, ultimate flexibility can be achieved, avoiding the need to recalculate a new UDC or returning to the original equation containing "e." For example, computing the activity remaining after 10 min of decay time using the UDC and a simple unit CF on a calculator with a y^x function is as follows:

$$A_0 = 10\ mCi;\ UDC = 0.89;\ Unit = 1\ hr$$

$$CF = 1\ hr/60\ min.$$

Thus:

$$A(10\ min) = A_0 \times (0.89)^{\#\,of\,hr\,units} \qquad \text{Eq. 13}$$

$$\begin{aligned} \#\ of\ hr\ units &= 10\ min \times CF \qquad \text{Eq. 14} \\ \#\ of\ hr\ units &= 10/60\ of\ an\ hr\ unit \end{aligned}$$

$$\begin{aligned} A\ (10\ min) &= A_0 \times (0.89)^{10/60} \\ A\ (10\ min) &= 10\ mCi \times (0.89)^{0.166} \qquad \text{Eq. 15} \\ A\ (10\ min) &= 10\ mCi \times 0.98 \\ A\ (10\ min) &= 9.8\ mCi. \end{aligned}$$

Suppose that the decay at 1.2 wk is needed, then the CF would be 24 hr/day × 7 days/wk:

$$CF = 168\ hr/wk;\ 1.2 \times 168 = 201.6\ units.$$

Thus:

$$\begin{aligned} A\ (1.2\ wk) &= 10\ mCi \times (0.89)^{(168\ hr/wk) \times 1.2\,wk} \\ A\ (1.2\ wk) &= 10\ mCi \times (0.89)^{201.6} \qquad \text{Eq. 16} \\ A\ (1.2\ wk) &= 6.27 \times 10^{-10}\ mCi. \end{aligned}$$

Application to Photon Attenuation

Another event that follows the mathematics of exponential loss is the attentuation of photons as they pass through matter (Table 4). This phenomena is appreciated, for example, in determining glomerular filtration rate (GFR) by the Gates' method, which takes into account the loss of photons relative to kidney depth in soft tissue (*3*). Likewise, photons are attenuated in the same manner when passed through various shielding devices.

Modification of the Gates' method using the UDC for soft-tissue attenuation of ^{99m}Tc activity yields the following equations:

Original equation:

Gates Method GFR

$$= \frac{\dfrac{Rt.\ k\ cts - bkg}{e^{-uxR}} + \dfrac{Lt.\ k\ cts - bkg}{e^{-uxL}}}{Pre\text{-}injection\ cts - Post\text{-}injection\ cts} \times 100\% \times 9.81270 - 6.8251.$$

TABLE 4. Parallel Photon Attenuation Problem

How much activity is transmitted through 3.5 cm of soft tissue when imaging a ^{99m}Tc source containing 20 mCi of activity? ($\mu = 0.1508$; $D_{1/2} = 4.59$ cm)

Traditional	UDC
$A_{(x)} = A_0 e^{-\mu x}$	UDC (^{99m}Tc for s.t.) = 0.86/cm
$A(3.5\ cm) = 20\ mCi \times e^{-(0.693/4.59) \times 3.5}$	$A(X) = A_0 \times (UDC)^{3.5}$
$A(3.5\ cm) = 20\ mCi \times e^{-0.15 \times 3.5}$	$A(3.5\ cm) = 20\ mCi \times (0.86)^{3.5}$
$A(3.5\ cm) = 20\ mCi \times e^{-0.528}$	$A(3.5\ cm) = 20\ mCi \times 0.59$
$A(3.5\ cm) = 20\ mCi \times 1/e^{0.528}$	$A(3.5\ cm) = 11.8\ mCi$
$A(3.5\ cm) = 20\ mCi \times 1/1.69$	
$A(3.5\ cm) = 20\ mCi \times 0.59$	
$A(3.5\ cm) = 11.8\ mCi$	

Kidney depth (xR = right kidney and xL = left kidney) in cms is determined by patient height (ht) in cms and patient weight (wt) in kgs:

$$xR = \frac{wt}{ht}\,3.3 + 0.7 \quad xL = \frac{wt}{ht}\,3.2 + 0.7.$$

Modified equation using the UDC = 0.86/cm:

Gates Method GFR

$$= \frac{\dfrac{\text{Rt. k cts} - \text{bkg}}{0.86^{\,xR}} + \dfrac{\text{Lt. k cts} - \text{bkg}}{0.86^{xL}}}{\text{Pre-injection cts} - \text{Post-injection cts}} \times 100\% \times 9.81270 - 6.8251.$$

SUMMARY AND CONCLUSIONS

The use of the UDC aids in an intuitive grasp of exponential mathematics, and allows for rapid calculations to be made in the clinical setting. The UDC and its applications can significantly simplify all single-termed exponential calculations. The selection of the unit from which the UDC is derived is dependent only on the useful time, distance, or "other measured" quantity that is commonly encountered. With the use of appropriate conversion factors, and a calculator with a y^x function, there is infinite flexibility, allowing any unit substitution to be performed without the need for complex exponential mathematics.

REFERENCES

1. Sorenson JA, Phelps ME. *Physics in Nuclear Medicine.* 2nd edition. Orlando: Grune and Stratton, Inc.; 1987:38–55.
2. Chandra R. *Introductory Physics of Nuclear Medicine.* 3rd edition. Philadelphia: Lea and Febiger; 1987:26–35.
3. Gates GF. Split renal function testing using Tc-99m DTPA. A rapid technique for determining differential glomerular filtration. *Clin Nucl Med* 1983;8:400–407.